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# Graph value for cooperative games

## Working paper

# GRAPH VALUE FOR COOPERATIVE GAMES

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ABSTRACT. We suppose that players in a cooperative game are located within a graph structure, such as a social network or supply route, that limits coalition formation to coalitions along connected paths within the graph. This leads to a generalisation of the Shapley value that is studied here from an axiomatic perspective. The resulting 'graph value' is endogenously asymmetric, with the automorphism group of the graph playing a crucial role in determining the relative values of players.

**Keywords:** Shapley value, network games.
**JEL classification:** D46, D72.

## 1. INTRODUCTION

The standard interpretation of the Shapley value, as a measure of the average marginal contribution of a player to each and every possible coalition, may strain credulity if taken literally in a great many social situations. This holds particularly when players may, due to affinity, consanguinity or other factors, have clear preferences for joining certain coalitions as opposed to others. Consider, for just one example, a job market. Is it not more likely that a potential hire will join a company if he knows someone within the company? How likely is it for a job seeker to join a company if she does not share a common language with any of its current employees?

Cases in which many theoretically possible coalitions will not realistically be formed are not limited to social situations alone. If one is studying cooperative coalitions amongst players connected via supply routes, computer networks or web links, there are clear structural reasons for entirely excluding some coalitions that would otherwise play a role in the calculation of the classic Shapley value and including in consideration instead only coalitions that are connected along the underlying network.

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Networks, for obvious reasons, have increasingly been a focus of study in several disciplines over the past two decades.[1] What we propose here is introducing network or graph structures directly into the study of coalitional game theory, by limiting consideration of potential coalitions solely to coalitions that are connected along the graph. Doing so, in the tradition of measuring average marginal contributions, yields different values that we argue may be more appropriate for assessing the values of players in many situations than the classic Shapley value.

This requires departing in some ways from the classic model of transferable utility games, which associates a certain worth to *every* coalition. That model implicitly assumes that the only force that drives the formation of coalitions is the worth they generate. The model we introduce here takes into account a proximity relation between players represented as edges of an undirected graph (a symmetric binary relation). It is assumed that a player only joins a coalition if he is connected to one of its members. As a result the only admissible coalitions are the connected subgraphs.

For our axioms we conservatively adopt the standard Shapley axioms (plus monotonicity), with minor adjustments to fit them for our model. The most significant difference this imposes is on symmetry (which is usually regarded as the least controversial of the Shapley value axioms). Classic symmetry cannot be carried over to our setting because the graph structure, and the relative positioning of players along the graph structure, is in itself an asymmetry. This leads to a weaker form of symmetry with respect only to automorphisms of the underlying graph.

By hewing close to most of the standard Shapley axioms, we are able to carry out a step-by-step development of concepts that are directly analogous to those associated with the standard Shapley value, such as probabilistic values and random values. The price of using a weaker symmetry axiom, however, is that it leads to a graph value that is not uniquely determined by the axioms; we instead derive a convex set of possible values. Specification of a unique graph value, it turns out, will in most cases require specifying a particular random ordering, intuitively corresponding to agreement amongst the players as to how coalitions are likely to be formed along the connected paths of the underlying graph.

On the other hand, the value we derive *is* a generalisation of the Shapley value, because when the underlying graph is the complete graph the set of admissible coalitions is again the full power set of the set of players. In that case, there is a unique graph value that is exactly the classic Shapley

[1] Perhaps a contemporary canonical example would be an on-line social network, with coalitions naturally growing in size by way of adding at each stage friends of current members.

value. Conversely, we show that there are graphs for which the graph value is unique and yet it is different from the Shapley value.

### 1.1. **Related Literature.**

Our main inspiration, and the paper that is most similar in approach to this one, is Álvarez, Hellman and Winter (2012), which proposes a way to measure the relative power of political parties in a parliament by explicitly taking into account a political spectrum. That paper notes that it is highly unlikely for a left-wing party to form a coalition with a party holding strongly diametrical right-wing views unless there are other parties in the coalition that can 'bridge' the ideological differences. In more general terms, a political party will tend to join a pre-existing coalition only if the coalition contains at least one other party that is ideologically close to it. To formalise this idea, Álvarez, Hellman and Winter (2012) postulates that parties can be ordered along a political spectrum (i.e., a strict linear ordering), from right to left, and a coalition will form only if it consists of a consecutive range of ideological views along this spectrum.[2]

One possible shortcoming of that approach is that it may be artificial to ascribe all ideological differences to positioning along a single linear ordering. In practice, ideologies are often multidimensional, relating to several issues. That observation led to the model presented in this paper, which is a generalisation of the model in Álvarez, Hellman and Winter (2012). As an added benefit, by extending the underlying topology of the connections between players to any graph, the model here is potentially applicable to a very wide range of cooperative situations, including but by no means restricted to political-coalitional settings.

Weakening the axiom of symmetry for the sake of considering variations on the Shapley value is a very old idea. Weighted Shapley values were proposed by Lloyd Shapley himself in his seminal PhD thesis (Shapley (1953b)). Each weighted Shapley value associates a positive weight with each player. These weights are the proportions of the players' shares in unanimity games. The symmetric Shapley value is the special case in which all weights are the same. This concept was studied axiomatically in Kalai and Samet (1987).

The weights in these models, however, are imposed exogenously, representing some pre-existing measure of the relative strengths of the players which is then used for calculating weighted Shapley values. In contrast, in

[2] As here, Álvarez, Hellman and Winter (2012) work with a weak version of symmetry and hence do not derive a unique value from the standard Shapley axioms alone. In that paper, an axiom reminiscent of balanced contributions axioms, relating to unanimity games, needs to be added to attain uniqueness of the value.

the approach here asymmetries arise endogenously from the positioning of the players along the underlying graph structure.

This paper is also far from the first to study situations in which not every coalition is feasible or equally likely. The issue is usually tackled by considering some structure on the set of players that circumscribes the way players can form coalitions. Games with these kind of structures are usually denoted games with restricted cooperation.

Among the earliest efforts in this direction, the beginnings of a large literature, are Aumann and Dreze (1975) and Owen (1977). These start from the supposition that cooperative games are endowed with a coalitional structure, an exogenously given partition of the players. When coalitions are formed, the players interact at two levels: first, bargaining takes place among the unions and then bargaining takes place inside each union. Within each union, however, every possible coalition is admissible.

Graphs appear explicitly in Myerson (1977), but in a very different role from the one they have in this paper. There, an undirected graph describes communication possibilities between the players. A modification of the Shapley value is then proposed under the assumption that coalitions that are not connected in this graph are split into connected components. In that model too, within components all possible coalitions are admissible.

1.2. **Content.** Section 2 defines the model and the basic concepts of coalitional games with an underlying graph. Section 3 provides an axiomatic definition for graph values and related solution concepts. Section 4 investigates a few special cases. Section 5 discusses the necessity of the axioms as well as a few questions for future research.

## 2. Graphs and Values

### 2.1. **Definitions.**

A finite set of *players* $N$, of cardinality $n = |N|$, will be assumed fixed throughout. We denote the set of all permutations over $N$, meaning bijective mappings $\pi : N \to N$, by $\Pi_N$. The $i$-th element of a permutation $\pi \in \Pi_N$ will be denoted by $\pi_i$, and we will also denote $\pi^{\swarrow i} := \{\pi_j \mid j < \pi^{-1}(i)\}$, i.e. the predecessors of $i$ in the list $\pi_1, \pi_2, \ldots, \pi_n$.

With tolerable abuse of notation, given a permutation $\pi \in \Pi_N$ we will also consider $\pi$ to be a mapping $\pi : 2^N \to 2^N$ by defining $\pi(\{i_1, i_2, \ldots, i_k\}) = \{\pi(i_1), \pi(i_2), \ldots, \pi(i_k)\}$. We will also abuse notation by sometimes writing $i$ instead of the singleton set $\{i\}$ when no confusion is possible, for the sake of readability.

The set of *coalitions* is the set of subsets of $N$. Conventionally, a *coalitional game over $N$* is given by a *characteristic function* $v$ which is a real-valued function over the set of all coalitions, i.e., $v : 2^N = \{S : S \subseteq N\} \to \mathbb{R}$ with the convention that $v(\emptyset) = 0$. Denote the set of all coalitional games by $\mathcal{K}$.

A *value for player $i$* on $\mathcal{K}$ is a function $\varphi_i : \mathcal{K} \to \mathbb{R}$. A *(group) value* on $\mathcal{K}$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, associates a vector in $\mathbb{R}^N$ with each game.

We next suppose that there is additional structure on $N$ making it a graph. An *undirected graph $G$* over $N$ is an ordered pair $(N, E)$, where $N$, the set of players, is now considered to be a set of vertices and $E$, the set of edges, is a set of pairs of distinct elements in $N$. A path is a sequence of edges connecting a sequence of vertices. Two players $i, j \in N$ are connected if $G$ contains a path from $i$ to $j$.

We will assume that every graph $G = (N, E)$ in this paper is connected, meaning that every pair of players are connected by some path.

The set of connected sub-graphs of a graph $G$ (including the empty set) will be denoted by $\mathcal{A}(G)$. Clearly, since $\mathcal{A}(G) \subseteq 2^N$, each element of $\mathcal{A}(G)$ is in particular a coalition. We will term $\mathcal{A}(G)$ the set of *admissible coalitions*.

For each player $i \in N$, denote

$$\mathcal{A}(G)^{-i} := \{S \in \mathcal{A}(G) \mid i \notin S \text{ and } S \cup i \in \mathcal{A}(G)\}. \tag{1}$$

$\mathcal{A}(G)^{-i}$ is always non-empty, because at minimum it contains the empty set. In addition, given an admissible coalition $T \in \mathcal{A}(G)$, denote

$$T^+ := \{i \in N \setminus T \mid T \cup i \in \mathcal{A}(G)\} \tag{2}$$

(hence in particular $N^+ = \emptyset$), and

$$T^- := \{i \in T \mid T \setminus i \in \mathcal{A}(G)\} \tag{3}$$

**Definition 1.** A characteristic function $v$ over the set of admissible coalitions, i.e., $v : \mathcal{A}(G) \to \mathbb{R}$, with the convention that $v(\emptyset) = 0$, is a *coalitional game over $G$*.

Denote the family of all coalitional games over a fixed set of players $N$ by $\mathcal{G}(N)$. We will frequently write simply $\mathcal{G}$ when $N$ is clear by context.

**Definition 2.** A sequence of distinct admissible coalitions

$$S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_k$$

ordered by containment is a *chain* over $G$.

A maximally ordered sequence of admissible coalitions

$$\emptyset = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_{|N|} = N$$

is a *maximal chain* over $G$.

The set of all maximal chains over $G$ will be denoted $\mathcal{C}(G)$. Note that for every pair of successive integers $k$ and $k+1$ in a maximal chain, by maximality $S_{k+1} \setminus S_k$ is a singleton. This leads to the following concept.

**Definition 3.** For each maximal chain $c \in \mathcal{C}(G)$ there is an *admissible permutation* of the elements of $N$, given by the mapping $d : \mathcal{C}(G) \to \Pi_N$ defined by:

$$d(c) = (S_1 \setminus S_0, S_2 \setminus S_1, \ldots, S_n \setminus S_{n-1}) \tag{4}$$

The mapping $d$ is obviously bijective. We henceforth denote $\mathcal{D}(G) := d(\mathcal{C}(G))$.

We adapt the following standard concepts from the literature on coalitional games. A game $v$ is *simple* if for every admissible coalition $S$, either $v(S) = 1$ or $v(S) = 0$. A game $v$ is *monotonic* if $v(S) \geq v(T)$ for all $S, T \in \mathcal{A}(G)$ satisfying $S \supseteq T$.

**Definition 4.** The *unanimity game* with *carrier* $T \in \mathcal{A}(G)$ is the monotonic simple game $U_T$ satisfying the condition that $U_T(S) = 1$ if and only if $T \subseteq S$.

Relatedly, as in Weber (1988), define $\widehat{U}_T$ to be the monotonic simple game satisfying the condition that $\widehat{U}_T(S) = 1$ if and only if $T \subsetneq S$. ♦

Following the lines of many standard proofs in the theory of coalitional games, it is easy to show that $\mathcal{G}(N)$ is a vector space of dimension $|\mathcal{A}(G)|$.

We also introduce here a non-standard concept:

**Definition 5.** Let $\Psi \subseteq \mathcal{A}(G)$. The *non-monotonic simple game* $W_\Psi$ with *multi-carrier* $\Psi$ is

$$W_\Psi(S) = \begin{cases} 1 & \text{if } S \in \Psi \\ 0 & \text{otherwise.} \end{cases}$$

♦

**Definition 6.** Over the family of games $\mathcal{K}$, it is standard to define a probabilistic value for player $i$ to be a value satisfying $\varphi_i(v) = \sum_{S \subseteq N \setminus i} p^i_S(v(S \cup i) - v(S))$ for a probability distribution $\{p^i_S\}_{S \subseteq N \setminus i}$. Over $\mathcal{G}$ the analogous expression for a *probabilistic value* is

$$\varphi_i(v) = \sum_{T \in \mathcal{A}(G)^{-i}} p^i_T(v(T \cup i) - v(T))$$

for a probability distribution $\{p^i_T\}$ over the set $\{T \in \mathcal{A}(G)^{-i}\}$. ♦

For a fixed game $v$, a player $i \in N$ is a *null player* if $v(S \cup \{i\}) = v(S)$ for all $S \in \mathcal{A}(G)^{-i}$. A player $i$ is a dummy player if $v(S \cup i) = v(S) + v(\{i\})$ for all $S \in \mathcal{A}(G)^{-i}$. A null player is a dummy player with $v(\{i\}) = 0$.

Let $\pi \in \Pi_N$ be a permutation. For every chain $c = (\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots \subsetneq G)$, the image of $c$ under $\pi$ trivially satisfies the condition that $\emptyset \subsetneq \pi(S_1) \subsetneq \pi(S_2) \subsetneq \ldots \subsetneq G$. There is no guarantee, however, that $\pi(S_k)$ is an admissible coalition for any particular $k < |N|$ the coalition (in the terminology introduced in Dubey and Weber (1977), the class of games $\mathcal{G}$ is not symmetric under all possible permutations). We will want to note when a permutation of a graph preserves admissible coalitions.

**Definition 7.** A permutation $\pi \in \Pi_N$ is an *automorphism* of $G$ if $\pi(S) \in \mathcal{A}(G)$ for all $S \in \mathcal{A}(G)$. ♦

Denote the set of automorphisms of $G$ by $Aut(G)$. Automorphisms are exactly what they are supposed to be, namely permutations of the graph structure:

**Lemma 1.** *A permutation $\pi$ is an* automorphism *of $G = (N, E)$ if and only if for every pair $i, j \in N$, $(i, j) \in E$ implies that $(\pi(i), \pi(j)) \in E$.*

**Proof**. In one direction, suppose that $\sigma$ is an automorphism and let $S = \{i, j\}$ be an admissible coalition of size two, which can only hold if $(i, j) \in E$. Then $\pi(S) = \{\pi(i), \pi(j)\}$ is also an admissible coalition. But that can only be true if $\pi(i)$ and $\pi(j)$ are connected, i.e., $(\pi(i), \pi(j)) \in E$.

In the other direction, first note that every permutation $\pi$ trivially maps the empty set and singleton sets to admissible coalitions. Suppose that $(i, j) \in E$ implies that $(\pi(i), \pi(j)) \in E$. Then all admissible coalitions of size two are mapped to admissible coalitions. From here proceed by induction: is $S$ is an admissible coalition of size $k$, then the assumption implies that every admissible coalition $S \cup i$ is mapped to an admissible coalition $\pi(S) \cup \pi(i)$. ∎

It follows immediately that for any automorphism $\pi \in Aut(G)$, for all players $i$, $\pi(S) \in \mathcal{A}(G)^{-\pi(i)}$ for each $S \in \mathcal{A}(G)^{-i}$ and $\pi(S^+) = \pi(S)^+$ for all $S \in \mathcal{A}(G)$ such that $S \neq N$. Furthermore, for every chain $\emptyset \subsetneq S_1 \subsetneq \ldots \subsetneq G$ the image $\emptyset \subsetneq \pi(S_1) \subsetneq \ldots \subsetneq G$ is also a chain in $\mathcal{C}(G)$. In the sequel we will consider $Aut(G)$ to be a group acting on $\mathcal{C}(G)$ or $\mathcal{D}(G)$.

**Example 1.** Let the set of edges $E$ be the set of all pairs of elements in $N$, i.e. the resulting graph $G = (N, E)$ is a complete graph. Then trivially every subset of $N$ is an admissible coalition of $G$ and every permutation is an automorphism. The standard Shapley value is a value (in fact, the unique value) on $\mathcal{G}$, the set of games over complete graphs.

**Example 2.** Enumerate the members of $N$ as $1, \dots, n$. Define the set of edges to be $E = \{(k, k+1) \mid 1 \leq k \leq n-1\}$. Call the resulting graph $G = (N, E)$ a spectrum graph. In this case the set of automorphisms contains only two elements: the identity mapping and the mapping that reverses the ordering of the players (so that player 1 is mapped to player $n$, player 2 to player $n-1$ and so on).

This structure and a related value over it is studied in Álvarez, Hellman and Winter (2012) ⧫

Strictly speaking, we need to distinguish between values for player $i$ on $\mathcal{K}$ and values on $\mathcal{G}$, because games on $\mathcal{K}$ are distinct from $\mathcal{G}$ (their domains are difference, because they admit different admissible coalitions), but we will usually refer simply to values without specifying the domain when the intended meaning is clear.

## 3. Axiomatics

**Linearity Axiom**. A value $\varphi_i$ for $i$ satisfies linearity if it is a linear function, i.e., for every pair of games $v, w \in \mathcal{G}$ and $\alpha \in \mathbb{R}$

$$\varphi_i(v + \alpha w) = \varphi_i(v) + \alpha\varphi_i(w).$$

A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ satisfies linearity if each of its individual constituent values does.

**Dummy Axiom**. A value $\varphi_i$ satisfies the dummy axiom if $\varphi_i(v) = v(i)$ whenever $i$ is a dummy player in any $v \in \mathcal{G}$. A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ satisfies the dummy axiom if each of its individual constituent values does.

**Monotonicity Axiom**. A value $\varphi_i$ satisfies the monotonicity axiom if $\varphi_i(v) \geq 0$ for every monotonic game $v \in \mathcal{G}$. A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ satisfies the monotonicity axiom if each of its individual constituent values does.

**Lemma 2.** *Let $\varphi_i$ be a value for $i$ on $\mathcal{G}$ satisfying linearity. Then there is a collection of constants $\{a_T\}_{T \in \mathcal{A}(G)}$ such that for all $v \in \mathcal{G}$*

$$\varphi_i(v) = \sum_{T \in \mathcal{A}(G)} a_T v(T).$$

*Furthermore, if $\varphi_i$ satisfies the dummy axiom and $i \notin T^+ \cup T^-$ then $a_T = 0$.*

**Proof**. Consider the game $W_{\{T\}}$ that assigns 1 to the coalition $T$ and 0 to all other coalitions. Any game $v$ can be written as $v = \sum_{T \in \mathcal{A}(G)} v(T) W_{\{T\}}$ and by linearity $\varphi_i(v) = \sum_{T \in \mathcal{A}(G)} \varphi_i(W_{\{T\}}) v(T)$. The proof is concluded by setting $a_T = \varphi_i(W_{\{T\}})$ and noting that $i \notin T^+ \cup T^-$ implies that $i$ is a null player of $W_{\{T\}}$. ∎

**Lemma 3.** *Let $\varphi_i$ be a value for $i$ on $\mathcal{G}$ satisfying linearity and the dummy and monotonicity axioms. Then $\varphi_i$ is a probabilistic value.*

**Proof**. Let $T \in \mathcal{A}(G)^{-i}$ be non-empty. Consider the multi-carrier game $W_{\{T,T\cup i\}}$. We first claim that player $i$ is a null player in $W_{\{T,T\cup i\}}$. To see this, consider first

$$W_{\{T,T\cup i\}}(T \cup i) = W_{\{T,T\cup i\}}(T) = 1.$$

Next, for any $S \in \mathcal{A}(G)^{-i} \setminus \{T\}$, one has

$$W_{\{T,T\cup i\}}(S \cup i) = W_{\{T,T\cup i\}}(S) = 0.$$

By the assumption of linearity, Lemma 2 implies that

$$\begin{aligned} 0 &= \varphi_i(W_{\{T,T\cup i\}}) \\ &= \sum_{S\in\mathcal{A}(G)} a_S W_{\{T,T\cup i\}}(S) \\ &= a_T W_{\{T,T\cup i\}}(T) + a_{T\cup i} W_{\{T,T\cup i\}}(T \cup i) \\ &= a_T + a_{T\cup i}. \end{aligned}$$

It follows that $a_T + a_{T\cup i} = 0$ for all non-empty $T \in \mathcal{A}(G)^{-i}$. For each such $T$ define $p^i_T = a_{T\cup i} = -a_T$. Then, by Lemma 2, for every game $v$

$$\varphi_i(v) = \sum_{\substack{T\in\mathcal{A}(G):\\ i\in T^+\cup T^-}} a_T v(T) + a_T = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T(v(T \cup i) - v(T)). \tag{5}$$

Next, consider the unanimity game $U_{\{i\}}$. Since player $i$ is a dummy player[3] of this game, $\varphi_i(U_{\{i\}}) = U_{\{i\}}(i) = 1$. In addition, $U_{\{i\}}(T \cup i) - U_{\{i\}}(T) = 1$ for all $T \in \mathcal{A}(G)^{-i}$. Equation (5) now enables us to deduce that

$$1 = \varphi_i(U_{\{i\}}) = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T(U_{\{i\}}(T \cup i) - U_{\{i\}}(T)) = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T.$$

All that remains is showing that $p^i_T \geq 0$ for all $T \in \mathcal{A}(G)^{-i}$. This follows from Equation (5), monotonicity and the fact that the game $\widehat{U}_T$ is monotonic, hence $p^i_T = \varphi_i(\widehat{U}_T) \geq 0$ ∎

**Efficiency Axiom**. A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies efficiency if for every $v \in \mathcal{G}$

$$\sum_{i\in N} \varphi_i(v) = v(N).$$

[3] We assume here the dummy axiom rather than the weaker null-player axiom.

**Lemma 4.** *Let $\varphi$ be a group value satisfying efficiency such that $\varphi_i(v) = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T(v(T\cup i)-v(T))$ for all $v \in \mathcal{G}$ and all $i \in N$. Let $T^+$ and $T^-$ be as defined in Equations (2) and (3). Then $\sum_{i\in N^-} p^i_{N\setminus i} = 1$. In addition, for all $T \in \mathcal{A}(G)$ such that $T \neq N$, $\sum_{i\in T^-} p^i_{T\setminus i} = \sum_{j\in T^+} p^j_T$.*

**Proof**.

We work here with the game $W_{\{T\}}$ that assigns 1 to the coalition $T$ and 0 to all other coalitions. Let $\varphi_N(v) = \sum_{i\in N}\varphi_i(v)$ for any $v \in \mathcal{G}$. It is straightforward to show that

$$\begin{aligned}\varphi_N(v) &= \sum_{i\in N}\sum_{T\in\mathcal{A}(G)^{-i}} p^i_T(v(T\cup i)-v(T)) \\ &= \sum_{T\in\mathcal{A}(G)} v(T)\left[\sum_{i\in T^-} p^i_{T\setminus i} - \sum_{j\in T^+} p^j_T\right].\end{aligned}$$

It immediately follows that $W_{\{T\}}(N) = \sum_{i\in T^-} p^i_{T\setminus i} - \sum_{j\in T^+} p^j_T$. But $W_{\{N\}}(N) = 1$, hence $\sum_{i\in N^-} p^i_{N\setminus i} = 1$, while $W_{\{T\}}(N) = 0$ for all $T \in \mathcal{A}(G)\setminus\{N\}$, hence $\sum_{i\in T^-} p^i_{T\setminus i} = \sum_{j\in T^+} p^j_T$. ∎

Let $\{r_\omega\}_{\omega\in\Pi_N}$ be a probability distribution over $\Pi_N$. For $\mathcal{K}$, a random order group value $\zeta = (\zeta_1,\ldots,\zeta_n)$ is defined by

$$\zeta_i(v) = \sum_{\omega\in\Pi_N} r_\omega(v(\omega^{\swarrow i}\cup i) - v(\omega^{\swarrow i})),$$

for all $i \in N$ and $v \in \mathcal{K}$.

The usual interpretation of this definition is that each permutation represents an ordered queue of the players, who enter a room one by one according to their number in the queue. Each permutation defines a dynamic way of forming a coalition, which grows by one player at a time, thus enabling us to measure the contribution of each player to the coalition formed by the players who preceded him or her in entering the room.

The corresponding notion here is that not every queue of entering players if possible: only those in which the next player to enter the room is 'connected' to at least one player who is already in the room are admissible. Hence we limit consideration only to admissible permutations, i.e. in the set $\mathcal{D}(G)$. In particular, now letting $\{r_\pi\}_{\pi\in\mathcal{D}(G)}$ be a probability distribution over $\mathcal{D}(G)$, a *random order (group) value* $\zeta = (\zeta_1,\ldots,\zeta_n)$ over $\mathcal{G}$ is defined by

$$\zeta_i(v) = \sum_{\pi\in\mathcal{D}(G)} r_\pi(v(\pi^{\swarrow i}\cup i) - v(\pi^{\swarrow i})),$$

for all $i \in N$ and $v \in \mathcal{G}$.

The most intuitive way to construct a random order value is to suppose that given an admissible coalition $S$ one has a conditional distribution over the players in $S^+$ that represents the probability of choosing the next player to join. This induces a probability distribution over all admissible permutations from which the weights of a random order value can be derived. Conversely, it is easy to calculate the conditional probability of a player joining an already-formed coalition from the weights of a random order value. Lemma 5 shows that probabilistic values can be derived from random order values. Lemma 6 shows that if the linearity, efficiency, monotonicity and dummy axioms are assumed then essentially every random order value is derived from a conditional probability measuring how likely a player is to join an already-formed coalition, with the random order values and probabilistic values derivable each from the other.

**Lemma 5.** *Let $(r_\pi)$ be a probability distribution over $\mathcal{D}(G)$.*[4] *Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be the associated random order value. Then there is a collection of probabilistic values $\varphi = (\varphi_1, \ldots, \varphi_n)$ such that $\varphi_i(v) = \zeta_i(v)$ for all $i \in N$ and all $v \in \mathcal{G}$.*

**Proof**. For $i \in N$ and $v \in \mathcal{G}$

$$\begin{aligned}
\zeta_i(v) &= \sum_{\pi \in \mathcal{D}(G)} r_\pi(v(\pi^{\swarrow i} \cup i) - v(\pi^{\swarrow i})) \\
&= \sum_{T \in \mathcal{A}(G)^{-i}} \left( \sum_{\{\pi \in \mathcal{D}(G) | \pi^i = T\}} r_\pi \right) (v(T \cup i) - v(T))
\end{aligned}$$

Setting $p^i_T = \sum_{\{\pi \in \mathcal{D}(G) | \pi^{\swarrow i} = T\}} r_\pi$ for all $i \in N$ and $T \in \mathcal{A}(G)^{-i}$ and using that to construct a collection of probabilistic values suffices to complete the proof. ∎

**Lemma 6.** *Let $\varphi$ be a group value satisfying the linearity, dummy, monotonicity and efficiency axioms. Then there is a random order value $\zeta = (\zeta_1, \ldots, \zeta_n)$ on $\mathcal{G}$ such that $\zeta_i(v) = \varphi_i(v)$ for all $i \in N$ and $v \in \mathcal{G}$.*

**Proof**. By Lemma 3, the values that satisfy the linearity, dummy, monotonicity and efficiency axioms are exactly the probabilistic values with weights

[4] Recalling that $\mathcal{D}(G)$ is the set $d(\mathcal{C}(G))$, the admissible permutations.

$\{p^i_T\}_{i\in N,\, T\in\mathcal{A}(G)^{-i}}$ satisfying

$$
\begin{aligned}
p^i_T &\geq 0 && \forall i \in N,\, T \in \mathcal{A}(G)^{-i},\\
\sum_{T\in\mathcal{A}(G)^{-i}} p^i_T &= 1 && \forall i \in N,\\
\sum_{i\in T^-} p^i_T &= \sum_{i\in T^+} p^i_T && \forall T \in \mathcal{A}(G)\setminus N,\\
\sum_{i\in N^-} p^i_N &= 1.
\end{aligned}
\tag{6}
$$

Consider the directed acyclic graph $D$ whose vertices are the admissible coalitions and edges are all ordered pairs of the form $(T, T\cup i)$. The conditions of (6) specify the set of flows of capacity 1 from $\emptyset$ to $N$ over $D$. This is a convex compact polytope in $\mathbb{R}^{E(D)}$, hence it is the convex hull of its extreme points. The extreme points are flows supported on a single path, and the latter correspond to (deterministic) random order graph values. ∎

**Symmetry Axiom**. A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ satisfies symmetry if

$$\varphi_i(v) = \varphi_{\pi(i)}(\pi \circ v)$$

for all $v \in \mathcal{G}$, all $\pi \in Aut(G)$ and all $i \in N$, where $\pi \circ v(T) := v(\pi^{-1}(T))$.

**Definition 8.** A measure $\mu \in \Delta(\mathcal{C}(G))$ is $Aut(G)$ *invariant* if $\mu(c) = \mu(\pi(c))$ for all $c \in \mathcal{C}(G)$ and all $\pi \in Aut(G)$.

**Definition 9.** A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ over a graph $G = (N, E)$ is a *graph value* if it satisfies linearity and the dummy, monotonicity, efficiency and symmetry axioms.

**Theorem 1.** *A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ over $G$ is a graph value if and only if it is a random order value whose weights are $Aut(G)$ invariant.*

**Proof**. In one direction, let $\varphi$ be a random order value whose weights are $Aut(G)$ invariant. It is easy to check that all the axioms are satisfied by such a value.

In the other direction, suppose that $\varphi$ satisfies the above axioms. By Lemma 3, linearity and the dummy and monotonicity axioms justify using probabilistic values, while Lemma 6 shows that adding efficiency implies that $\varphi$ is a random order value.

It remains to take symmetry into account. Let $i \in N$, $T_1 \in \mathcal{A}(G)^{-i}$, and $\pi \in Aut(G)$. Since $\pi \in Aut(G)$, $\pi(T_1) \in \mathcal{A}(G)^{-\pi(i)}$. We have that $p^i_{T_1} = \varphi_i(\widehat{U}_{T_1})$ and $p^{\pi(i)}_{\pi(T_1)} = \varphi_{\pi(i)}(\widehat{U}_{\pi(T_1)})$. By symmetry, $\varphi_i(\widehat{U}_{T_1}) = \varphi_{\pi(i)}(\widehat{U}_{\pi(T_1)})$, hence $p^i_{T_1} = p^{\pi(i)}_{\pi(T_1)}$. Going back to the random order weights from the probabilistic values yields $Aut(G)$ invariant weights. ∎

Since $Aut(G)$ is a group acting on the set $\mathcal{D}(G)$ (equivalently, on $\mathcal{C}(G)$) we can consider the set $\mathcal{O}(G)$ of orbits of $Aut(G)$. The set $\mathcal{O}(G)$ partitions $\mathcal{D}(G)$. Hence we can choose a representative element from each orbit $\omega \in \mathcal{O}(G)$. The condition of $Aut(G)$ invariance of random order weights immediately implies the next two corollaries.

**Corollary 1.** *A group value over $G$ is a graph value if and only if its associated random order value with weights $(r_\pi)_{\pi\in\mathcal{D}(G)}$ satisfies the condition that there exists a collection of non-negative weights $\{\rho_\omega\}_{\omega\in\mathcal{O}(G)}$ with $\sum_{\omega\in\mathcal{O}(G)} \rho_\omega = 1$ such that $r_\pi = \rho_\omega$ for each $\omega \in \mathcal{O}(G)$ and each $\pi \in \omega$.*

**Corollary 2.** *For each orbit $\omega \in \mathcal{O}(G)$ denote by $U(\omega)$ the uniform probability distribution over $\{\pi\}_{\pi\in\omega}$. A group value over $G$ is a graph value if and only if the system of weights of the associated random order value is contained in the convex hull of $\{U(\omega)\}_{\omega\in\mathcal{O}(G)}$.*

## 4. Shapley Value vs Graph Value

Let $G$ be the complete graph over $N$, as in Example 1. Since $Aut(G) = \Pi_N$ in this case, there is only one orbit, and Corollary 1 implies that there exists a unique graph value. This unique graph value is precisely the Shapley value. It is, of course, a celebrated result of Shapley (1953a) that the Shapley value is unique, but we see this emerging from our discussion here from the perspective of graph values.

In contrast to the Shapley value, the graph value in general is not unique, because there may be several orbits. A graph is *entirely anti-symmetric* if $|\mathcal{O}(G)| = |\mathcal{D}(G)|$. This occurs, for example, if $Aut(G)$ consists solely of the identity permutation; there are many well-known examples of such graphs. If $G$ is a entirely anti-symmetric graph then any probability distribution over $\mathcal{D}(G)$ defines the weights of a random-order value that is a graph value for $G$. It follows from this that there are graphs whose set of graph values contains more than one point.

From previous results it is clear that if $|\mathcal{O}(G)| = 1$ then there is only one graph value, namely the one random-order value that assigns uniform weight to each element of $\mathcal{D}(G)$. This, however does not mean that $|\mathcal{O}(G)| = 1$ is a necessary condition for the existence of a unique graph value, as the next result shows.

An *$n$-cycle*, for $n \geq 3$ is the graph whose vertex set is $\{1, \dots, n\}$ with edge set $E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, 1\}\}$. There are two orbits for each

$n$-cycle, which may be termed the 'cycle-structure order preserving orbit' and the 'non-order preserving orbit'.

**Claim 1.** *The graph value over the $n$-cycle is unique for all $n$. The graph value over the 3-cycle is the Shapley value, but for all $n \geq 4$ the graph value differs from the Shapley value.*

**Proof.** Let $G$ be an $n$-cycle . If $n = 3$ the $n$-cycle is the complete graph over 3 elements and therefore there is a unique graph value that is the Shapley value. We concentrate henceforth on the case $n \geq 4$ and and let $\varphi$ be any graph value over $G$.

By construction, for each player $i$ there are exactly two players $j$ and $k$ that are connected to $i$ in $G$. Let $T \subset G$ be a connected coalition of players in $G$ of size $1 < |T| < n$. Define $i$ to be an internal vertex of $T$ if each of the two players $j$ and $k$ connected to $i$ are also in $T$. Consider the unanimity game $U_T$. If $i$ is an internal vertex of $T$ then $i$ is pivotal with respect to a given admissible permutation $\pi$ of $N$ iff $i$ is the last player in the ordering defined by $\pi$. By symmetry, each internal player has an equal probability of being last; it follows that $\varphi_i(U_T) = 1/n$ for all internal vertices $i$.

The two players on the boundary of $T$ are symmetric and they must therefore receive the same value by the symmetry axiom. By efficiency,

$$\varphi_j(U_T) = \frac{1}{2}\left(1 - \frac{|T| - 2}{n}\right).$$

for each player $j$ on the boundary of $T$.

This is sufficient to show that the graph value is unique over the $n$-cycle and that it differs from the Shapley value, which would give each player equal value over $U_T$, not distinguishing between interal and boundary players. ∎

Finally, we consider one more example of a graph with an interesting graph value.

**Example 3.** The *$n$-star graph* is defined over the vertex set $\{0, i_1, i_2, \ldots, i_n\}$ with edges $\{\{0, i_1\}, \{0, i_2\}, \ldots, \{0, i_n\}\}$. Consider the simple majority game $v$ and any graph value $\psi$ over the $n$-star graph. Then straightforward combinatorial calculations show that

$$\psi_0(v) = 0$$
$$\psi_{i_1}(v) = \psi_{i_2}(v) = \ldots = \psi_{i_n}(v) = \frac{1}{n}$$

♦

The result in Example 3 is again very different from the Shapley value, because the internal vertex receives a zero value under all circumstances. points to a weakness of the graph value. This is because the graph value essentially counts the number of times each player is a pivot player among all admissible permutations. In the simple majority game over the star graph, the internal node can never be the pivot player in any admissible coalition.

This may at first seem surprising, since one natural representation of the internal node of a star graph is a market maker through whom everyone else needs to go through to conduct trade, or similarly a hub for resource distribution. One might think this would grant the internal player a great deal of power, yet the axioms that we assumed, which are almost verbatim adaptations of the standard Shapley axioms for our setting in which only connected coalitions may be formed, end up giving that player zero value.

One explanation for this phenomenon is as follows. In the standard Shapley value approach, measuring the average marginal gain a player causes by joining coalitions is entirely equivalent to measuring the average marginal loss he causes by leaving coalitions. In the graph value setting, this equivalence no longer obtains. Since only connected coalitions may be formed, leaving a coalition is only possible if the remaining coalition is connected. Another way to put this ides is that if a market maker disconnects from the other players, then no coalition of more than one player can be formed. A market maker who quits cannot improve his own payoff.

## 5. Axioms Redundancy and Solution Uniqueness

The original Shapley axioms include additivity, null player, efficiency and symmetry. In the definition of graph values additivity and the null player axioms are replaced with the stronger assumptions of linearity and the dummy axiom, and the monotonicity axiom is added. Naturally, we would like to know if the (seemingly) weaker set of axioms implies the stronger set of axioms.

It is not too hard to show that additivity, efficiency, monotonicity[5] and the null player axioms imply linearity and the dummy axiom; therefore the following questions arise.

**Question 1.** What are the graphs for which any solution concept that satisfies linearity, efficiency, symmetry and the dummy axiom is monotonic?

**Question 2.** What are the graphs for which any solution concept that satisfies additivity, efficiency, symmetry and the dummy axiom is linear?

[5] Continuity, a weaker assumption than monotonicity, is sufficient.

The complete graph satisfies the conditions of the above questions, as it yields the standard Shapley value. It is unknown whether or not it is the *only* graph that satisfies these conditions. It turns out that uniqueness of the solution, however, is necessary in order to satisfy those conditions.

**Proposition 1.** *For every graph for which there exist two distinct graph values, there exists a solution that satisfy linearity, efficiency, symmetry and the dummy axiom, but not monotonicity.*

**Proof.** The set of solutions that satisfy linearity, efficiency, symmetry and the dummy axiom is closed under affine combinations. The set of (monotonic) graph values is compact; therefore if it is not a point it is strictly contained in its affine span. ∎

**Proposition 2.** *For every graph for which there exist two distinct solutions that satisfy linearity, efficiency, symmetry and the dummy axiom, there exists a solution that satisfies additivity, efficiency, symmetry and the dummy axiom, but not linearity.*

**Proof.** Linear functions of Euclidean spaces are continuous and any solution that satisfies additivity and the null-player axiom is linear over $\mathbb{Q}$; therefore we must find a discontinuous solution. Suppose there are two distinct solutions $\varphi$ and $\xi$. If any of them is discontinuous then we are done. Otherwise, we have an open ball of games on which $|\varphi_i - \xi_i| \geq \epsilon$, for some player $i$ and $\epsilon > 0$. Take a countable set of $\mathbb{Q}$-linearly independent games in that ball,$\{v_n\}_{n=1}^{\infty}$, and complete it to a basis of $\mathcal{G}$ over $\mathbb{Q}$, $\mathcal{B}$. Define a $\mathbb{Q}$-linear solution $\zeta$, by

$$\begin{aligned} \zeta(v_n) &= \varphi(v_n) + n(\varphi(v_n) - \xi(v_n)), \\ \zeta(v) &= \varphi(v_n) && \text{for } v \in \mathcal{B} \setminus \{v_n\}_{n=1}^{\infty}. \end{aligned}$$

The solution $\zeta$ satisfies additivity, efficiency, symmetry and the dummy axiom, but $\zeta_i$ diverges on any convergent subsequence of $(v_n)_{n=1}^{\infty}$. ∎

The converse of Propositions 1 and 2 is unknown, that is, it is not known whether or not uniqueness determines the answer to Questions 1 and 2.

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